EQUATIONS OF THE THIRD AND FOURTH DEGREE

Algebra among the ancients

Algebra is a science almost entirely due to the moderns. I say almost entirely, for we have one treatise from the Greeks, that of **Diophantus,** who flourished in the third century of the Christian era. This work is the only one which we owe to the ancients in this branch of mathematics. When I speak of the ancients I speak of the Greeks only, for the Romans have left nothing in the sciences, and to all appearances did nothing.

Algebra in Europe

Be that as it may, the Europeans, having received algebra from the Arabs, were in possession of it one hundred years before the work of **Diophantus** was known to them. They made, however, no progress beyond equations of the first and second degree. In the work of **Paciolus,** the general resolution of equations of the second degree, such as we now have it, was not given. We find in this work simply rules, expressed in bad Latin verses, for resolving each particular case according to the different combinations of the signs of the terms of equation, and even these rules applied only to the case where the roots were real and positive. Negative roots were still regarded as meaningless and superfluous. It was geometry really that suggested to us the use of negative quantities, and herein consists one of the greatest advantages that have resulted from the application of algebra to geometry – a step which we owe to **Descartes.**

Tartaglia (1500-1559) – Cardan (1501-1576)

In the subsequent period the resolution of *equations of the third degree* was investigated and the discovery for a particular case ultimately made by a mathematician of Bologna named **Scipio Ferreus** (1515). Two other Italian mathematicians, **Tartaglia** and **Cardan,** subsequently perfected the solution of Ferreus and rendered it general for all equations of the third degree. At this period, Italy, which was the cradle of algebra in Europe, was still almost the sole cultivator of the science, and it was not until about the middle of the sixteenth century that treatise on algebra began to appear in France, Germany, and other countries. The works of **Peletier** and **Buteo** were the first which France produced in this science, the treatise of the former having been printed in 1554 and that of the latter in 1559.

Tartaglia expounded his solution in bad Italian verses in a work treating of divers' questions and inventions printed in 1546, a work which enjoys the distinction of being one of the first to treat of modern fortifications by bastions.

About the same time (1545) Cardan published his treatise *Ars Magna,* or *Algebra,* in which he left scarcely anything to be desired in the resolution of equations of the third degree. Cardan was the first to perceive that equations had several roots and to distinguish them into positive and negative. But he is particularly known for having first remarked the so called *irreducible case* in which the expression of the real roots appears in an imaginary form. Cardan convinced himself from several special cases in which the equation had rational divisors that the imaginary form did not prevent the roots from having a real value. But it remained to be proved that not only were the roots real in the irreducible case, but that it was impossible for all three together to be real except in that case. This proof was afterwards supplied by **Vieta,** and particularly by **Albert Girard,** from considerations touching the trisection of an angle.

The irreducible case

We shall revert later on to the *irreducible case of equations of the third degree,* not solely because it presents a new form of algebraic expressions which have found extensive application in analysis, but because it is constantly giving rise to unprofitable inquiries with a view to reducing the imaginary form to a real form and because it thus presents in algebra a problem which may be placed upon the same footing with the famous problems of the duplication of the cube and the squaring of the circle in geometry.

The mathematicians of the period under discussion were wont to propound to one another problems for solution. These problems were in the nature of public challenges and served to excite and to maintain in the minds of thinkers that fermentation which is necessary for the pursuit of science. The challenges in question were continued down to the beginning of the eighteenth century by the foremost mathematicians of Europe, and really did not cease until the rise of the Academies which fulfilled the same end in a manner even more conducive to the progress of science, partly by the union of the knowledge of their various members, partly by the intercourse which they maintained between them, and not least by the publication of their memoirs, which served to disseminate the new discoveries and observations among all persons interested in science.

Biquadrate equations

The challenges of which we speak supplied in a measure the lack of Academies, which were not yet in existence, and we owe to these passages at arms many important discoveries in analysis. Such was the resolution of *equations of the fourth degree,* which was propounded in the following problem.

Ferrari Bombelli (1522-1565)

According to **Bombelli,** of whom we shall speak again, **Louis Ferrari** of Bologna resolved the problem by a highly ingenious method, which consists in dividing the equation into two parts both of which permit of the extraction of the square root. To do this it is necessary to add to the two numbers quantities whose determination depends on an equation of the third degree, so that the resolution of equations of the fourth degree depends upon the resolution of equations of the third and is therefore subject to the same drawbacks of the irreducible case.

The *Algebra* of Bombelli was printed in Bologna in 1579 in the Italian language. It contains not only the discovery of Ferrari but also divers other important remarks on equations of the second and third degree and particularly on the theory of radicals by means of which the author succeeded in several cases in extracting the imaginary cube roots of the two binomials of the formula of the third degree in the irreducible case, so finding a perfectly real result and furnishing thus the most direct proof possible of the reality of this species of expressions.

Such is a succinct history of the first progress of algebra in Italy. The solution of equations of the third and fourth degree was quickly accomplished. But the successive efforts of mathematicians for over two centuries have not succeeded in surmounting the difficulties of the equation of the fifth degree.

Theory of equations

Yet these efforts are far from having been in vain. They have given rise to the many beautiful theorems which we possess on the formation of equations, on the character and signs of the roots, on the transformation of a given equation into others of which the roots may be formed at pleasure from the roots of the given equation, and finally, to the beautiful considerations concerning the metaphysics of the resolution of equations from which the most direct method of arriving at their solution, when possible, has resulted. All this has been presented to you in previous lectures and would leave nothing to be desired if it were but applicable to the resolution of equations of higher degree.

Vieta and **Descartes** in France, **Harriot** in England, and **Hudde** in Holland, were the first after the Italians whom we have just mentioned to perfect the theory of equations, and since their time there is scarcely a mathematician of note that has not applied himself to its investigation, so that in its present state this theory is the result of so many different inquiries that it is difficult in the extreme to assign the author of each of the numerous discoveries which constitute it.

I promised to revert to the irreducible case. To this end it will be necessary to recall the method which seems to have led to the original resolution of equations of the third degree and which is still employed in the majority of the treatises on algebra.

Equations of the third degree

Let us consider the general equation of the third degree deprived of its second term, which can always be removed; in a word, let us consider the equation $x^3 + px + q = 0$

Suppose $x = y + z$ where y and z are two new unknown quantities, of which one consequently may be taken at pleasure and determined as we think most convenient. Substituting this value for x we obtain *the transformed equation*

$$y^3 + 3y^2z + 3yz^2 + z^2 + p(y + z) + q = 0$$

The reduced equation

Factoring the two terms $3y^2z + 3yz^2$ we get $3yz(y + z)$ and the transformed equation may be written as follows:

$$y^3 + z^3 + (3yz + p)(y + z) + q = 0$$

Putting the factor multiplying $y + z$ equal to zero – which is permissible owing to the two undetermined quantities involved – we shall have the two equations $3yz + p = 0$ and $y^3 + z^3 + q = 0$ from which y and z can be determined. The means which most naturally suggests itself to this end is to take from the first equation the value of z

$$z = -\frac{p}{3y}$$

and to substitute it in the second equation, removing the fractions by multiplication. So proceeding, we obtain the following equation of the sixth degree in y called *the reduced equation*

$$y^6 + qy^3 - \frac{p^3}{27} = 0$$

which, since it contains two powers only of the unknown quantity of which one is the square of the other, is resolvable after the manner of equations of the second degree and gives immediately

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

from which, by extracting the cube root, we get

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and finally $$x = y + z = y - \frac{p}{3y}$$

This expression for x may be simplified by remarking that the product of y by the radical $\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ supposing all the quantities under the sign to be multiplied together, is

$$\sqrt[3]{-\frac{p^3}{27}} = -\frac{p}{3}$$

The term $\frac{p}{3y}$ accordingly, takes the form

$$-\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and we have

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

an expression in which the square root underneath the cubic radical occurs in both its plus and minus forms and where consequently there can, on this score, be no occasion for ambiguity.

Cardan's rule

This last expression is known as *the* ***Rule of Cardan,*** and there has hitherto been no method devised for the resolution of equations of the third degree which does not lead to it. Since cubic radicals naturally present but a single value, it was long thought that Cardan's rule could give but one of the roots of the equation, and that in order to find the two others we must have recourse to the original equation and divide it by $x-\alpha$ α being the first root found. The resulting quotient being an equation of the second degree may be resolved in the usual manner. The division in question is not only always possible, but it is also very easy to perform. For in the case we are considering the equation being $x^3+px+q=0$

If α is one of the roots we shall have $\alpha^3+p\alpha+q=0$ which subtracted from the preceding will give $x^3-\alpha^3+p(x-\alpha)=0$ a quantity divisible by $x-\alpha$ and having as its resulting quotient

$$x^2+\alpha x+\alpha^2+p=0$$

CHAPTER X

SOLUTION OF NUMERICAL EQUATIONS

1. Newton's Method

To find the root between 2 and 3 of $x^3-2x-5=0$ **Newton** replaced x by $2+p$ and obtained $p^3+6p^2+10p-1=0$ Since p is a decimal, he neglected the first two terms and set $10p-1=0$ so that $p=0.1$ approximately. Replacing p by $0.1+q$ in the preceding cubic equation, he obtained $q^3+6.3q^2+11.23q+0.061=0$ Dividing -0.061 by 11.23 he obtained -0.0054 as the approximate value of q Neglecting q^3 and replacing q by $-0.0054+r$ he obtained the following equation: $6.3r^2+11.16196r+0.000541708=0$

Dropping $6.3r^2$ he found r and hence we have the following equation: $x=2+0.1-0.0054-0.00004853=2.09455147$

This value is in fact correct to the seventh decimal place. But the method will not often lead as quickly to so accurate a value of the root.

The method is usually presented in the following form. Given that α is an approximate value of a real root of $f(x)=0$ we can usually find a nearer approximation $\alpha+h$ to the root by neglecting the powers h^2, h^3 of the small number h in Taylor's formula

$$f(\alpha+h)=f(\alpha)+f'(\alpha)h+f''(\alpha)\frac{h^2}{2}+\ldots$$

and hence by taking

$$f(\alpha)+f'(\alpha)h=0\ ,\ h=\frac{-f(\alpha)}{f'(\alpha)}$$

We then repeat the process with $\alpha+h$ in place of the former α

Thus in Newton's example, we have, for $\alpha=2$

$$h=\frac{-f(2)}{f'(2)}=\frac{1}{10}\ ,\ \alpha'=\alpha+h=2.1$$

$$h'=\frac{-f(2.1)}{f'(2.1)}=\frac{-0.061}{11.23}=-0.0054\ldots$$

2. Graphical Discussion of Newton's Method

Using rectangular coordinates, consider the graph of $y=f(x)$ and the point P on it with the abscissa $OQ=\alpha$ (Fig.1). Let the tangent at P meet the x-axis at T and let the graph meet the x-axis at S Take $h=QT$ the subtangent.

Fig.1 Fig.2

Then $QP=f(\alpha)$ $f'(\alpha)=\tan XTP=-f(\alpha)/h$

$$h=\frac{-f(\alpha)}{f'(\alpha)}$$

In the fictitious graph in Figure 1, $OT=\alpha+h$ is a better approximation to the root OS than $OQ=\alpha$ The next step (indicated by dotted lines) gives a still better approximation OT_1

If, however, we had begun with the abscissa α of a point P_1 near a bend point, the subtangent would be very large and the method would probably fail to give a better approximation. Failure is certain if we use a point P_2 such that a single bend point lies between it and S

We are concerned with the approximation to a root previously isolated as the only real root between two given numbers α and β These should be chosen so nearly equal that $f'(x)=0$ no bend point between α and β Further, if $f'(x)=0$ has a root between our limits, our graph will have an inflexion point with an abscissa between α and β, and the method likely will fail (Fig.2).

Let, therefore, neither $f'(x)$ nor $f''(x)=0$ vanish between α and β Since f'' preserves its sign in the interval from α to β while f changes in sign, f'' and f will have the same sign for one end point. According to the abscissa of this point is α or β we take $\alpha=\alpha$ or $\alpha=\beta$ for the first step of Newton's process. In fact, the tangent at one of the end points meets the x-axis at a point T with an abscissa within the interval from α to β If $f'(x)$ is positive in the interval, we have Fig.3 or Fig.4 If f' is negative we have fig.5 or fig.6.

Fig.3 fig.4 fig.5

In Newton's example, the graph between the point with the abscissas $\alpha=2$ and $\beta=3$ is of the type in Fig.3, but more nearly like a vertical straight line. In view of this feature of the graph, we may safely take $\alpha=\alpha$ as did Newton, although our general procedure would be to take $\alpha = \beta$ The next step, however, accords with our present process; we have $\alpha = 2$ $\beta = 2.1$ in Fig.3 and hence we now take $\alpha = \beta$ getting

$$\frac{0.061}{11.23} = 0.0054$$

as the subtangent, and hence $2.1 - 0.0054$ as the approximate root.

If we have secured (as in Fig.3 or Fig.5) a better upper limit to the root than β we may take the abscissa c of the intersection of the chord AB with the x-axis as a better lower limit than α By similar triangles, we have

$$-f(\alpha) : c - \alpha = f(\beta) : \beta - c$$

$$c = \frac{\alpha f(\beta) - \beta f(\alpha)}{f(\beta) - f(\alpha)} \qquad \textbf{(1)}$$

This method of finding the value of c intermediate to α and β is called the *method of interpolation (regula falsi).*

In Newton's example $\alpha = 2$, $\beta = 2.1$ we take

$$f(\alpha) = -1 \text{ , } f(\beta) = 0.061 \text{ , } c = 2.0942$$

The advantage of having c at each step is that we know a close limit of the error made in the approximation to the root.

We may combine the various possible cases discussed into one: if $f(x) = 0$ has a single real root and $f'(x) = 0$, $f''(x) = 0$ have no real root between α and β, and if we designate by β that one of the numbers α and β for which $f(\beta)$ and $f''(\beta)$ have the same sign, then the root lies in the narrower interval from c to $\beta - f(\beta)/f'(\beta)$ where c is given by **(1).**

It is possible to prove this theorem algebraically and to show that by repeated application of it we can obtain two limits α', β' between which the root lies, such that $\alpha' - \beta'$ is numerically less than any assigned positive number. Hence the root can be found in this manner to any desired accuracy.

Example: $f(x) = x^3 - 2x^2 - 2$, $\alpha = 2\frac{1}{4}$, $\beta = 2\frac{1}{2}$

$$\text{Then } f(\alpha) = -\frac{4}{6}.\frac{7}{4} \text{ , } f(\beta) = \frac{2}{8}$$

Neither of the roots 0 , $\frac{4}{3}$ of $f'(x) = 0$ lies between α and β so that $f(x) = 0$ has a single real root between these limits (Ch. IX § 1). Nor is the root $\frac{2}{3}$ of $f''(x) = 0$ within these limits. The conditions of the theorem are therefore satisfied. For $\alpha \prec x \prec \beta$ the graph is of the type in Fig.3. We find that

$$c = \frac{5}{2}\frac{5}{3}\frac{9}{8} = 2.307 \text{ , } \beta' = \beta - \frac{f(\beta)}{f'(\beta)} = 2.3714$$

$$\beta' - \frac{f(\beta')}{f'(\beta')} = 2.3597$$

For $x = 2.3593$, $f(x) = -0.00003$ We therefore have the root to four decimal places. For $\alpha = 2.3593$ $\quad f'(\alpha) = 7.2620, \alpha - \frac{f(\alpha)}{f'(\alpha)} = 2.3593041$ which is the value of the root correct to 7 decimal places. For, if we change the final digit from 1 to 2, the result is greater than the root in view of our work, while if we change it to 0, $f(x)$ is negative.

Pages 113, 114.

Exercises

(Preserve the numerical work for later use)

1. Find the root between 1 and 2

2.

3.

Solutions

NEWTON'S METHOD OF APPROXIMATION

122. When we know an approximate value of a root, we may easily obtain other values of it, more and more exact, by a method invented by **Newton,** which rapidly attains its object. We shall give this method, first in the form in which it was proposed by its author, and afterwards with the conditions which **Fourier** has shown to be necessary for its complete success.

Let $f(x)=0$ be an equation having a root c between α and b the difference of these limits $b-\alpha$ being a small fraction whose square may be neglected in the process of approximation.

Let c_1 a quantity between α and b be assumed as the first approximation to c then $c=c_1+h$ where h is very small;

$$\therefore f(c_1+h)=0,$$

$$or\ f(c_1)+f'(c_1)h+f''(c_1)\frac{h^2}{2}+\ldots+h^n=0$$

Now since h is very small, $h^2,h^3,\&c.$ are very small compared with h; also none of the quantities $f''(c_1),f'''(c_1),\&c.$ can become very great, since they result from substituting a finite value in integral functions of x; therefore, provided $f'(c_1)$ be not very small (that is, provided $f'(x)=0$ have no root nearly equal to c_1 or to c and consequently $f(x)=0$ no other root nearly equal to c besides the one we are approximating to), all the terms in the series after the first two may be neglected in comparison with them; and we have, to determine h_1 the resulting approximate value of h the equation

$$f(c_1)+h_1f'(c_1)=0;$$

$$\therefore h_1=-\frac{f(c_1)}{f'(c_1)}=-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1}$$

and the second approximation is

$$c_2=c_1+h_1=c_1-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1}$$

Similarly, starting from c_2 instead of c_1 the third approximate value will be $c_3=c_2-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_2}$ and so on; and if we can be certain that each new value is nearer to the truth than the preceding, there is no limit to the accuracy which may be obtained.

Example 1

$$x^3 - 2x - 5 = 0$$

Here, one root lies between 2 and 3, and the equation can have only one positive root; also, upon narrowing the limits, we find that $x = 2$ gives a negative, and $x = 2.2$ a positive result, therefore 2.1 differs from the root by a quantity less than 0.1 and we may assume $c_1 = 2.1$ Hence we take: $c_2 = 2.1 - \left(\frac{x^3 - 2x - 5}{3x^2 - 2}\right)_{x=2.1} = 2.1 - \frac{0.061}{11.23}$,

$$or\ c_2 = 2.1 - 0.0054 = 2.0946$$

Similarly, $c_3 = 2.09455149$

Example 2

$$x^3 - 7x - 7 = 0$$

There is only one positive root, lying between 3 and 3.1; and it equals 3.048917339

OBS. To guard against over correction, that is, against applying such a correction to an approximate value, as shall make the new value differ more from the root by excess than the original approximate value did by defect, or vice versa, we must be certain that each new value is nearer to the truth than the preceding; this gives rise to the following conditions, first noticed by Fourier.

123. For the complete success of Newton's method of approximation, the following conditions are necessary.

(1) The limits between which the required root is known to lie must be so close, that no other root of $f(x) = 0$ and no root of $f'(x) = 0$ or $f''(x) = 0$ lies between them.

(2) The approximation must be begun and continued from that limit which makes $f(x)$ and $f''(x)$ have the same sign.

Let c be a root of $f(x) = 0$ which lies between α and b, $\alpha \prec b$, c_1 the first approximate value, and h the whole correction, so that

$c = c_1 + h$; then $f(c_1) + h = 0$, or $f(c_1) + hf'(\lambda) = 0$, or λ being some quantity between c_1 and c (Art. 121).

Therefore, supposing $\lambda = c_1$ which amounts to neglecting all powers of h above the first, and requires that $f(x) = 0$ have no root besides c in that interval, and calling the resulting approximate value of h, h_1 we have the following equation:

$$f(c_1) + h_1 f'(c_1) = 0$$

Now the true value is $c = c_1 + h$ that is the first approximate value is c_1 with error h and the second approximate value is $c_2 = c_1 + h_1$ with error $h - h_1$ which (neglecting signs) must be less than h, i.e. $h^2 - (h - h_1)^2$ must be positive, or $2hh_1 - h_1^2 = +$, or $\frac{h}{h_1} - \frac{1}{2} = +$, or $\frac{f'(c_1)}{f'(\lambda)} - \frac{1}{2} = +$ which condition (since λ is an intermediate quantity between c_1 and c or between α and b) cannot in all cases be secured unless $f'(x)$ be incapable of changing its sign between α and b; i.e. unless $f'(x) = 0$ have no roots between α and b

Moreover, we must have $\frac{f'(c_1)}{f'(\lambda)} \succ \frac{1}{2}$ or $\frac{f'(c_1)}{f'(\lambda)} \succ 1$

Now if $f''(x)$ preserve an invariable sign between α and b i.e. if $f''(x) = 0$ have no root in that interval, then $f'(x)$ will increase or diminish continually from α to b; therefore c_1 must be taken equal to that limit which gives $f'(x)$ its greatest numerical value without regard to sign.

First, let $f'(x)$ and $f''(x)$ have the same sign from α to b; then $f'(x)$ increases continually in that interval; therefore we must have $c_1 = b$ or we must begin from the greatest limit. But $f(b)$ has the same sign as $f(c+h) = f(c) + hf'(c) = hf'(c)$ or as $f'(c)$; therefore we must have c_1 equal to that limit which makes $f(x)$ and $f''(x)$ have the same sign.

Secondly, let $f'(x)$ and $f''(x)$ have contrary signs from α to b; then $f'(x)$ diminishes continually in that interval; therefore we must have $c_1 = \alpha$ or we must begin from the lesser limit. But $f(\alpha)$ has the same sign as $f(c-h) = f(c) - hf'(c) = -hf'(c)$ or as $-f'(c)$; therefore in this case, equally as in the former, we must have c_1 equal to that limit which makes $f(x)$ and $f''(x)$ have the same sign.

These conditions being fulfilled, we have

$$\frac{f'(c_1)}{f'(\lambda)} - 1 = + \text{ or } \frac{h - h_1}{h_1} = +$$

$$\text{or } \frac{c - c_2}{c_2 - c_1} = +;$$

therefore c_2 lies between c and c_1; hence the new limit c_2 fulfils the requisite conditions, and we may with certainty by it continue the approximation.

124. To estimate the rapidity of the approximation, we have the following table: error in first approximate value $c_1 = h$

error in second approximate value $c_2 = h - h_1$

$$f'(c_1) + hf'(c_1) + \frac{1}{2}h^2 f''(\mu) = 0$$

$$f(c_1) + h_1 f'(c_1) = 0;$$

$$\text{But } \therefore (h - h_1) f'(c_1) + \frac{1}{2}h^2 f''(\mu) = 0$$

$$or\ h - h_1 = -\frac{1}{2}h^2 \frac{f''(\mu)}{f'(c_1)}$$

Let the greatest value which $f''(x)$ can assume between a and b (which will be either $f''(\alpha)$ or $f''(b)$ if $f'''(x) = 0$ have no root in the interval) be divided by the least value of $2f'(x)$ in that interval which will be either $2f''(\alpha)$ or $2f''(b)$ and let the quotient be denoted by C;

then neglecting signs $h-h_1 \prec h^2C$; hence if the first error h in c_1 be a small decimal, the error $h-h_1$ with which c_2 is affected (since C will not, except in particular cases, be very large) will be very small compared with h and if the quantity C be less than unity, the number of exact decimals in the result will be doubled by each successive operation. The quantity C when thus computed for a given interval, preserves the same value throughout the operations which it may be necessary to make in order to approximate to the value of the root lying in the interval; and as we thus know a limit to the difference between the approximate value already found and the true value, we may always avoid calculating decimals which are inexact, and only obtain those which are necessarily correct.

Example

$$6x^3-141x+263=0$$

This equation has two positive roots, one between 2.7 and 2.8, and the other between 2.8 and 2.9 Now $f'(x)=18x^2-141=0$ has a root equal to $\sqrt{\frac{47}{6}}=2.798$ between 2.7 and 2.8 therefore these limits are not sufficiently close; but this root is greater than 2.79; also 2.7 and 2.79 substituted in $f(x)$ gives results with different signs; and 2.7 substituted in $f(x)$ and $f''(x)$ gives results with the same sign; therefore $c_1=2.7$

With regard to the other interval 2.8 2.9 $f'(x)=0$ $f''(x)=0$ have no roots between these limits, and 2□ makes $f(x)$ and $f''(x)$ have the same sign; therefore $c_1=2.9$ and starting from these values we are certain in each case to get a new value nearer to the truth.

Again, the greatest value which $\frac{f''(x)}{f'(2\square 7)}$ can assume in the interval 2.7 2.79 is nearly equal to 10 hence if h_1 h_2 be consecutive errors, we shall have $h_2 \prec \frac{1}{2}(h_1)^2 \square 10$

The same formula will be found to be true for consecutive errors in the interval 2.8 2.9

SOLUTION OF A QUADRATIC EQUATION

84. Let the equation be reduced to the form

$$x^2 + px + q = 0$$

Then this may be transformed into $y^2 = \alpha$ by taking away its second term. For, putting $x = y - \frac{1}{2}p$ (Art. 29), we have the following equation:

$$y^2 - py + \frac{p^2}{4} + py - \frac{p^2}{2} + q = 0 \ or \ y^2 = \frac{p^2}{4} - q$$

$$\therefore x = -\frac{p}{2} + y = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Hence, if $\frac{p^2}{4} \succ q$ the roots are real; if $\frac{p^2}{4} = q$ they are equal, and $x^2 + px + q = \left(x + \frac{1}{2}p\right)^2$ is the perfect square; if $\frac{p^2}{4} \prec q$ the roots are impossible. Also if α, β be the two roots:

$$\alpha + \beta = -p, \ \alpha\beta = q$$

$$\therefore (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 - 4q$$

Hence also any trinomial

$$\alpha x^2 + bx + c \ or \ \alpha\left(x^2 + \frac{b}{\alpha}x + \frac{c}{\alpha}\right)$$

will be resolvable into two real simple factors or not, according as $\frac{b^2}{4\alpha^2} \succ \frac{c}{\alpha}$ or $\frac{b^2}{4\alpha^2} \prec \frac{c}{\alpha}$ and it will be a perfect square when $\frac{b^2}{4\alpha^2} = \frac{c}{\alpha}$ or $b^2 = 4\alpha c$ i.e. when the square of the coefficient of the middle term is equal to four times the product of the coefficients of the extreme terms.

85. Any impossible expression of the form $\alpha \pm \beta\sqrt{-1}$ may be transformed into $r\left(\cos\theta \pm \sqrt{-1}\sin\theta\right)$

For α and β being real quantities, there always exists an angle θ such that $\tan\theta = \frac{\beta}{\alpha}$ then $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ $\sin\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ If therefore $\sqrt{\alpha^2+\beta^2} = r$ we have $\alpha \pm \beta\sqrt{-1} = r\left(\cos\theta \pm \sqrt{-1}\sin\theta\right)$

Hence any pair of imaginary roots of an equation may be represented by the formula $r\left(\cos\theta \pm \sqrt{-1}\sin\theta\right)$ and the quantity r which is equal to $\sqrt{\alpha^2+\beta^2}$ = square root of the product of the roots, and is always real, is called *the Modulus of the expression* $\alpha + \beta\sqrt{-1}$ and is that quantity by which the impossible roots are estimated, when, as is sometimes requisite, they are compared in regard to magnitude with the real roots.

In the case of the expression $-\frac{p}{2} \pm \sqrt{-1}\sqrt{q - \frac{p^2}{4}}$ which represents the roots of $x^2 + px + q = 0$ (Art. 84), $r = \sqrt{q}$ $\cos\theta = -\frac{p}{2\sqrt{q}}$

Hence $x^2 + px + q = x^2 - 2r\cos\theta x + r^2$ that is, any irreducible quadratic factor of an equation $x^2 + px + q$ where $\frac{p^2}{4} \prec q$ may be transformed into $x^2 - 2r\cos\theta x + r^2$ by making

$$r = \sqrt{q} \text{ and } \cos\theta = -\frac{p}{2\sqrt{q}}$$

86. To solve an equation of the form $x^{2n} + px^n + q = 0$

Putting $x^n = y$ we find $y^2 + py + q = 0$ If this have two real roots α and b then the $2n$ values of x are the roots of the equations $x^n - \alpha = 0$ and $x^n - b = 0$

If the roots of the quadratic are imaginary, i.e. if $\frac{p}{4} \prec q$ then making $r = \sqrt{q}$ and $\cos\theta = -\frac{p}{2\sqrt{q}}$ the proposed equation becomes

$$x^{2n} - 2r\cos\theta x^n + r^2 = 0$$

$$\text{or } x^{2n} - 2\cos\theta x^n + 1 = 0$$

changing x^n into $x^n r$ which has already been solved (p. 26).

SOLUTION OF A CUBIC EQUATION BY CARDAN'S RULE

87. Let the equation, by Art. 29, be reduced to the form

$$x^3 + qx + r = 0$$

and put $x = y + z$ that is suppose x equal to the sum of two other unknown quantities;

$$\therefore x^3 = 3yz(y+z) + y^3 + z^3$$

and therefore the proposed equation becomes

$$(3yz + q)(y+z) + y^3 + z^3 + r = 0$$

Now since we have two unknown quantities, and have made only one supposition respecting them, namely, that $y + z = x$ we are at liberty to make another; let therefore $3yz + q = 0$

$$\text{or } y^3 z^3 = -\left(\frac{q}{3}\right)^3 \quad \therefore y^3 + z^3 = -r$$

Hence y^3 z^3 are the roots of the equation

$$t^2 + rt - \left(\frac{q}{3}\right)^3 = 0$$

since the coefficient of the second term with its sign changed is equal to their sum, and the last term is equal to their product. Solving this equation, we get

$$t = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$$

$$\therefore y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \quad z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$$

$$\text{but } x = y + z$$

$$\therefore x=\left(-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right)^{\frac{1}{3}}$$

an expression which (since the cube root of any quantity has three values) contains implicitly the three roots; and the quantities q and r are either real or imaginary.

88. This method only serves for the numerical solution of cubic equations in those cases in which the equation has two impossible roots.

Let the coefficients q and r be real quantities, and let m and n be the arithmetical values of the two surds in the value of x and $1\ \alpha\ \alpha^2$ the three cube roots of unity; then the three values of y are (Art. 82) $m\ \alpha m\ \alpha^2 m$ and those of z are $n\ \alpha n\ \alpha^2 n$ By combining these values two and two to form $y+z$ we shall have nine values of x The number being tripled by reason of our having employed $y^3z^3=-\left(\frac{q}{3}\right)^3$ instead of $yz=-\frac{q}{3}$ the relation arising immediately in the process; and we observe that every combination will satisfy $y^3z^3=-\left(\frac{q}{3}\right)^3$ but only three the given condition $yz=-\frac{q}{3}$ which latter are the roots, viz.

$$m+n\quad \alpha m+\alpha^2 n\quad \alpha^2 m+\alpha n$$

or substituting for $\alpha\ \alpha^2$ their values $-\frac{1}{2}\left(1\pm\sqrt{-3}\right)$ (p. 17), the three roots are

$$m+n \text{ and } -\frac{1}{2}\left\{m+n\pm(m-n)\sqrt{-3}\right\}$$

Hence, as long as the expression $\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}$ is possible, the values of m and n are possible, and the equation has one possible root, the numerical value of which, as also those of the two imaginary roots, may be obtained after somewhat laborious calculations from the above formulae.

But when the expression $\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}$ is impossible, m and n are impossible, and all the three roots appear under imaginary forms; whereas, the equation, being of an odd degree, has at least one real root; and indeed, since $\frac{r^2}{4}+\frac{q^3}{27}$ is negative, it has (Art. 54) all its roots real; in this case therefore the above formulae, although algebraical expressions for the roots, cannot on account of the imaginary quantities which they involve, be applied to furnish the numerical values of the roots.

OBS. It is easily seen that the six superfluous values of x above mentioned would be the roots of the proposed equation, supposing q to be successively replaced by αq and $\alpha^2 q$ since each of the relations $yz=-\frac{1}{3}\alpha q$ and $yz=-\frac{1}{3}\alpha^2 q$ leads to $y^3z^3=-\frac{1}{27}q^3$

Example 1

$$x^3-3x-110=0$$

$$\frac{q}{3}=-1 \quad \frac{r}{2}=-55$$

$$\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}=\sqrt{(55)^2-1}=12\sqrt{21}=54.991$$

$$\therefore x=(55+54.991)^{\frac{1}{3}}+(55-54.991)^{\frac{1}{3}}=$$
$$=4.79\ldots+0.208\ldots=4.999\ldots=5$$

or

$$x=-\frac{1}{2}\left\{5\pm(4.582\ldots)\sqrt{-3}\right\}=-\frac{1}{2}\left(5\pm\sqrt{-62.99\ldots}\right)=$$
$$=-\frac{1}{2}\left(5\pm3\sqrt{-7}\right)$$

Example 2

$$x^3-12x-65=0$$

$$x=5 \text{ or } -\frac{1}{2}\left(5\pm3\sqrt{-3}\right)$$

Example 3

$$x^3 - 2x - 5 = 0$$

$x = 2.0945\ldots$ or $= -1.0472\ldots \pm (1.1362\ldots)\sqrt{-1}$

89. In the case of the roots being all real, which, for the reason just stated is called *the Irreducible Case,* that is, when q is negative, and $\frac{q^3}{27} \succ \frac{r^2}{4}$ it may be observed that the assumption in the process

$$y^3 + z^3 = -r \quad y^3 z^3 = +\left(\frac{q}{3}\right)^3$$

are inconsistent with one another; for the product of two real quantities can never exceed the square of half their sum. In this case we can show that, in the expressions for the roots, the impossible quantities destroy one another, and the three roots are real. For let the values of m^3 and n^3 be represented by $\alpha \pm b\sqrt{-1}$ Then expanding by the binomial theorem, and taking P and Q to denote real functions of α and b we have

$$\left(\alpha \pm b\sqrt{-1}\right)^{\frac{1}{3}} = P \pm Q\sqrt{-1}$$

$$\therefore m + n = 2P \quad m - n = 2Q\sqrt{-1}$$

and the three values of x are $2P$ and $-\frac{1}{2}\left(2P \pm 2Q\sqrt{3}\right)$ which are all real. This mode of proceeding, however, is useless in finding the numerical values of the roots; for if we convert $\left(\alpha + b\sqrt{-1}\right)^{\frac{1}{3}}$ into a series, P and Q will be expressed by series which rarely converge, and from which we can never obtain the exact values of P and Q; and if we attempt to express the cube root of $\alpha \pm b\sqrt{-1}$ by an expression of the same form, we shall have to solve a cubic of the same kind as the one in question.

Hence Cardan's rule succeeds for the following forms, where q and r are essentially positive,

$$x^3 + qx \pm r = 0 \text{ in all cases,}$$

$$x^3 - qx \pm r = 0 \text{ when } \frac{q^3}{27} \prec \frac{r^2}{4}$$

and it fails for $x^3 - qx \pm r = 0$ when $\frac{q^3}{27} \succ \frac{r^2}{4}$

all the roots of which are real.

TRIGONOMETRICAL EXPRESSIONS FOR THE ROOTS OF A CUBIC

By the employment of Trigonometrical formulae, we obtain simple expressions for the roots of a Cubic, not only in the irreducible case when the algebraical formulae cease to be of use for computing the numerical values of those roots, but in all the other cases.

92. If in the expression

$$\left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} = \sqrt{\frac{q}{3}}\left\{-\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}} \pm \sqrt{\frac{r^2}{4}\left(\frac{3}{q}\right)^3 + 1}\right\}^{\frac{1}{3}}$$

we put $\cos\phi = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$ it becomes $\sqrt{\frac{q}{3}}\left(-\cot\phi \pm \cos ec\phi\right)^{\frac{1}{3}}$

Hence, reducing the real root of $x^3 + qx + r = 0$ is $\sqrt{\frac{q}{3}}\left(\tan^{\frac{1}{3}}\frac{\phi}{2} - \cot^{\frac{1}{3}}\frac{\phi}{2}\right)$ which, by putting $\tan\frac{\phi}{2} = \tan^3\theta$ may be further transformed into $-2\sqrt{\frac{q}{3}}\cot 2\theta$

Similarly, the real root of $x^3 - qx + r = 0$, $\frac{q^3}{27} \prec \frac{r^2}{4}$ becomes (by putting $\cos ec\phi = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, $\tan\frac{\phi}{2} = \tan^3\theta$), $-2\sqrt{\frac{q}{3}}\cos ec 2\theta$

Also, in the irreducible case $x^3 - qx \pm r = 0$, $\frac{q^3}{27} \succ \frac{r^2}{4}$ the expression $\mp\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = \left(\frac{q}{3}\right)^{\frac{3}{2}}\left\{\mp\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}} \pm \sqrt{-1}\left(1 - \frac{27r^2}{4q^3}\right)^{\frac{1}{2}}\right\}$

by making $\cos\phi = \pm\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$ becomes equally

$$\left(\frac{q}{3}\right)^{\frac{3}{2}}\left(-\cos\phi \pm \sqrt{-1}\sin\phi\right) \text{ or } \left(\frac{q}{3}\right)^{\frac{3}{2}}\left\{\cos(\pi\pm\phi) \pm \sqrt{-1}\sin(\pi\pm\phi)\right\}$$

and therefore the three values of x are

$$-2\sqrt{\frac{q}{3}}\cos\frac{\phi}{3} \quad , \quad 2\sqrt{\frac{q}{3}}\cos\left(\frac{\pi\pm\phi}{3}\right)$$

SOLUTION OF A BIQUADRATE EQUATION BY DESCARTES'S METHOD

93. Let the proposed equation be reduced to the form

$$x^4 + qx^2 + rx + s = 0$$

and as the first member may be always regarded as the product of two real quadratic factors, we may assume it

$$= \left(x^2 + px + f\right)\left(x^2 - px + g\right)$$

$$= x^4 + \left(g + f - p^2\right)x^2 + (pg - pf)x + fg$$

(effecting the multiplication), where the coefficients of the second terms p and $-p$ are equal and of opposite signs, because the second term of the proposed equation is wanting, that is, the sum of its roots is zero. Hence, equating coefficients:

$$g + f - p^2 = q \quad pg - pf = r \quad fg = s$$

$$\text{or } g + f = q + p^2 \quad g - f = \frac{r}{p}$$

$$\therefore 2g = q + p^2 + \frac{r}{p} \quad 2f = q + p^2 - \frac{r}{p}$$

$$\therefore 4fg = q^2 + 2qp^2 + p^4 - \frac{r^2}{p^2} = 4s$$

$$\text{or } p^6 + 2qp^4 + \left(q^2 - 4s\right)p^2 - r^2 = 0$$

the equation for determining p which rises to the sixth degree, because a polynomial of four dimensions, may have (Art. 17) $\frac{4\cdot 3}{1\cdot 2}$ or six divisors of the second order. Also, because the values of p are the sums of every two roots of the proposed biquadrate, and because the sum of these roots is zero, and therefore the sum of any two is equal and of a contrary sign to the sum of the other two, therefore the values of p will be in pairs differing only in sign; this is the reason why the equation for determining p involves only even powers of p and may therefore be depressed to a cubic by putting $p^2 = y$ The reducing cubic is

$$y^3 + 2qy^2 + \left(q^2 - 4s\right)y - r^2 = 0$$

which (Art. 10) has necessarily one real positive root; let this be e^2 then the four values of x are contained in the quadratic equations

$$x^2 + ex + \frac{1}{2}\left(q + e^2 - \frac{r}{e}\right) = 0$$

$$x^2 - ex + \frac{1}{2}\left(q + e^2 + \frac{r}{e}\right) = 0$$

Example

$$x^4 - 3x^2 - 42x - 40 = 0$$

Here $q = -3 \quad r = -42 \quad s = -40$ and the reducing cubic is

$$y^3 - 6y^2 + 169y - (42)^2 = 0$$

which has a root equal to 9, (Art. 67), $\therefore e = 3 \quad \therefore x^2 + 3x + 10 = 0$ $x^2 - 3x - 4 = 0$ The roots of these quadratics $-\frac{1}{2}\left(3 \pm \sqrt{-31}\right)$, -1 , 4 are the roots of the proposed equation.

94. The reducing cubic will have all its roots real, unless two of the roots of the proposed biquadrate are possible and unequal, and two impossible.

For the square of the sum of any two roots of the proposed is a root of the reducing cubic; if therefore the proposed have all its roots real, the reducing cubic will have all its roots real; or if the proposed have all its roots imaginary, and therefore of the form

$$\alpha + \beta\sqrt{-1} \quad \alpha - \beta\sqrt{-1} \quad -\alpha + \beta'\sqrt{-1} \quad -\alpha - \beta'\sqrt{-1}$$

since their sum is zero, then the square of the sum of every two will be real, and therefore the cubic will have all its roots real. But if the proposed have two real unequal roots, and two imaginary roots, and therefore of the forms

$$\alpha+\beta\sqrt{-1} \quad \alpha-\beta\sqrt{-1} \quad -\alpha+\gamma \quad -\alpha-\gamma$$

then the square of the sum of a real and an imaginary root will be imaginary, and therefore the cubic will have one and consequently two imaginary roots. As it is only in the latter case that a numerical solution of the reducing cubic can be obtained, therefore **DesCartes's method** can only be applied to those cases in which two roots of the biquadrate are possible and unequal, and two are impossible.

It will be observed that in the latter case, if the two real roots are equal to one another, i.e. if $\gamma=0$ the cubic will have all its roots real; but as two of them are equal, it can still be solved.

95. If the roots of the reducing cubic can be obtained, and are put under the forms $(2\alpha)^2$ $(2\beta)^2$ $(2\gamma)^2$ then the four roots of the biquadrate are

$$-(\alpha+\beta+\gamma) \quad \beta+\gamma-\alpha \quad \alpha+\gamma-\beta \quad \alpha+\beta-\gamma$$

For, $-\frac{1}{2}q=\alpha^2+\beta^2+\gamma^2$ and $r^2=(8\alpha\beta\gamma)^2$

let $p^2=(2\alpha)^2$ or $p=\pm 2\alpha$

therefore, taking the upper sign,

$$f=\frac{1}{2}\left(q+p^2-\frac{r}{p}\right)=-\left(\alpha^2+\beta^2+\gamma^2\right)+2\alpha^2-2\beta\gamma=$$
$$=\alpha^2-(\beta+\gamma)^2$$

therefore the first reducing quadratic is $x^2+2\alpha x+\alpha^2-(\beta+\gamma)^2=0$ which gives for x the values $-(\alpha+\beta+\gamma)$ $\beta+\gamma-\alpha$ Similarly, the other quadratic, taking $p=-2\alpha$ is $x^2-2\alpha x+\alpha^2-(\beta-\gamma)^2=0$ which gives the other values $\alpha+\gamma-\beta$ $\alpha+\beta-\gamma$

Hence the roots of the biquadrate are symmetrical functions of the roots of the reducing cubic; and whatever root of the reducing cubic is used in the process, the same values of x are obtained.

SOLUTION OF A COMPLETE BIQUADRATE BY FERRARI'S METHOD

96. Let the equation be

$$x^4+px^3+qx^2+rx+s=0$$

and let it be supposed the same as $\left(x^2+\frac{p}{2}x+m\right)^2-(kx+l)^2=0$ where $k,\ l,\ m$ are unknown, and are to be determined so as to make the latter equation coincide with the proposed. Now we shall have the following equation

$$\left(x^2+\frac{p}{2}x+m\right)^2=x^4+px^3+\left(\frac{p^2}{4}+2m\right)x^2+pmx+m^2$$

$$-(kx+l)^2= \qquad -k^2x^2-2klx-l^2$$

therefore, by comparing this with the proposed, we have to determine $k,\ l,\ m$ the equations

$$\frac{p^2}{4}+2m-k^2=q \quad pm-2kl=r \quad m^2-l^2=s$$

Substituting in the second, the values of k and l obtained from the first and third, we get for the reducing cubic

$$8m^3-qm^2+(2pr-8s)m-p^2s+4qs-r^2=0\ldots \quad \mathbf{(1)}$$

which will necessarily give one real value for m Then k and l are known; and we find the two following reducing quadratics

$$x^2+\left(\frac{p}{2}+k\right)x+m+l=0$$

$$x^2+\left(\frac{p}{2}-k\right)x+m-l=0$$

Example

$$x^4+x^3-4x^2+1=0$$

The reducing cubic is $8m^3+16m^2-16m-33=0$ in which a value of $2m$ is -3 Therefore $k=\frac{1}{2}\sqrt{5}\quad l=\frac{1}{2}\sqrt{5}$ and the proposed equation is decomposed into

$$x^2+\frac{1}{2}\left(1+\sqrt{5}\right)x+\frac{1}{2}\left(-3+\sqrt{5}\right)=0$$

$$x^2+\frac{1}{2}\left(1-\sqrt{5}\right)x+\frac{1}{2}\left(-3+\sqrt{5}\right)=0$$

97. In this method the reducing cubic will have all its roots real, unless two roots of the biquadrate are possible and two impossible; for suppose the roots to be $\alpha,\beta,\ \gamma,\delta$ and suppose any two α,β to satisfy the first reducing quadratic, and consequently γ,δ the second:

$\therefore m+l=\alpha\beta\quad m-l=\gamma\delta\quad \therefore m=\frac{1}{2}\left(\alpha\beta+\gamma\delta\right)$ and the other values of m must be $\frac{1}{2}\left(\alpha\gamma+\beta\delta\right)\quad \frac{1}{2}\left(\alpha\delta+\beta\gamma\right)$

Hence if $\alpha,\beta,\gamma,\delta$ be either all possible, or all impossible, the values of m are real; but if two roots of the biquadrate be possible and two impossible, then two values of m will be impossible, and the reducing cubic may be solved by Cardan’s rule. In the latter case, however, if the two real roots are equal, the cubic will have all its roots real; but it may be solved, because two of them will be equal.

If in the equation **(1)** we substituted for $2m$ the value $k^2-\frac{1}{4}p^2+q$ it would be transformed into an equation of the sixth degree in k but containing only even powers of k and might be taken for the reducing equation; then since

$$\alpha+\beta=-\frac{1}{2}p-k\quad \gamma+\delta=-\frac{1}{2}p+k$$

therefore $-2k=\alpha+\beta-\gamma-\delta$ the expression for the roots of the reducing equation in k which is a linear function of the roots of the proposed, and can, by permuting those roots, assume six values, equal two and two, and of contrary sings.

SOLUTION OF A BIQUADRATE BY EULER'S METHOD

98. Let the equation be reduced to the form

$$x^4+qx^2+rx+s=0$$

and assume $x=y+z+u$

$\therefore x^2=y^2+z^2+u^2+2(yz+yu+zu)$ or $x^2-\left(y^2+z^2+u^2\right)=2(yz+yu+zu)$

$\therefore x^4-2x^2\left(y^2+z^2+u^2\right)+\left(y^2+z^2+u^2\right)^2=4\left(y^2z^2+y^2u^2+z^2u^2\right)$
$+8yzu(y+z+u)$

or, replacing $y+z+u$ by x and transposing

$$x^4-2x^2\left(y^2+z^2+u^2\right)-8yzux+\left(y^2+z^2+u^2\right)^2-4\left(y^2z^2+y^2u^2+z^2u^2\right)=0$$

In order that this may coincide with the proposed, we must have

$q=-2\left(y^2+z^2+u^2\right)$ $r=-8yzu$ $s=\left(y^2+z^2+u^2\right)^2-4\left(y^2z^2+y^2u^2+z^2u^2\right)$

or $y^2+z^2+u^2=-\dfrac{q}{2}$ $y^2z^2+y^2u^2+z^2u^2=\dfrac{q^2-4s}{16}$ $yzu=-\dfrac{r}{8}$

$$\text{or } y^2z^2u^2=\frac{r^2}{64}$$

hence y^2, z^2, u^2 are the roots of the cubic

$$t^3+\frac{q}{2}t^2+\frac{q^2-4s}{16}t-\frac{r^2}{64}=0$$

Let t^2, t'^2, t''^2 denote the three values of t in this equation;

$$\therefore y=\pm t \quad z=\pm t' \quad u=\pm t''$$

which six (6) values, combined three and three, would give eight (8) values of $y+z+u$ or x instead of four (4); the number being doubled because we have used $y^2z^2u^2=\dfrac{r^2}{64}$ instead of the given condition $yzu=-\dfrac{r}{8}$ which only allows those values of y, z, u to be combined which give, when multiplied together, a result with a contrary sign to r

Hence, if r be negative, there must be either two negative quantities, or none, in every combination; and if r be positive, there must be either two positive quantities, or none, in every combination representing a root. Therefore, in the former case, that is, when r is negative, the roots are

$$t-t'-t'' \quad t'-t-t'' \quad t''-t-t' \quad t+t'+t''$$

and in the latter case, when r is positive, the roots are

$$t+t'-t'' \quad t+t''-t' \quad t+t''-t \quad -t-t'-t''$$

and it will be observed, that the second set of roots results from the first, by changing the sign of any one of the quantities $t,\ t',\ t''$

99. In this case also, the reducing cubic will have all its roots real, except when the proposed has two (2) possible and two (2) impossible roots. Since the last term of the reducing cubic is essentially negative; it will always have one (1) real positive root t^2 and the remaining roots will be either both positive, both negative, or impossible; that is, of the forms $t'^2, t''^2 \ -t'^2, -t''^2$ or

$$\rho^2\left(\cos 2\theta \pm \sqrt{-1}\sin 2\theta\right)$$

Hence, whether the reducing cubic has three positive roots, two negative roots, or impossible roots, the biquadrate $x^4+qx^2-rx+s=0$ will have its roots respectively of the forms

$$t\pm(t'+t'') \quad -t\pm(t'-t'')$$

$$t\pm\sqrt{-1}(t'-t'') \quad -t\pm\sqrt{-1}(t'+t'')$$

$$t\pm 2\rho\cos\theta \quad -t\pm\sqrt{-1}\ 2\rho\sin\theta$$

In the case of the equation $x^4+qx^2-rx+s=0$ we must change the sign of t in the above expressions, and the results will be its roots.

If $2\theta=\pi$ or the two real roots of the biquadrate become equal, then, as before, the reducing cubic has three real roots, two of which are alike.

Lagrange's method of solving both cubic and biquadrate equations will be given in Section X. He shows that all the other methods, though different in appearance, ultimately involve the same principle; viz. that of making the solution of the proposed equation depend upon that of a reducing equation whose roots are linear functions of the roots of the proposed equation, and of the roots of unity of the same degree as the proposed equation. For equations of a higher degree than the fourth, the dimension of the reducing equation exceeds that of the proposed.

The solution of the general equation of the fifth degree is, as has been stated, impossible; but in Section VIII we shall show that it can be transformed, so as to want any three consecutive terms between the first and the last.

CHAPTER IV

APPROXIMATION TO THE ROOTS OF NUMERICAL EQUATIONS

57. Newton's Method of Approximation. This method is not as convenient in the solution of numerical equations involving algebraic functions as is the method of Horner, but it has the advantage of being applicable to numerical equations involving transcendental functions. For instance, **Newton's method** can be used in finding x in $x-\sin x=2$

Let $f(x)=0$ be the given equation. Suppose that we know a quantity α which differs from one of the values of x by the small quantity h Then we have $x=\alpha+h$ By Taylor's Theorem we shall have the following equation:

$$f(x)=f(\alpha+h)=f(\alpha)+hf'(\alpha)+\frac{h^2}{2}f''(\alpha)+\ldots$$

Since h is small, we get, by neglecting higher powers of h an approximate value of h from the equation $f(\alpha)+hf'(\alpha)=0$ namely $h=-\frac{f(\alpha)}{f'(\alpha)}$ We have approximately $x=\alpha-\frac{f(\alpha)}{f'(\alpha)}$ Letting this new approximation to the value of x be represented by b we may repeat the above process and secure a still closer approximation, and so on.

Example 1 Solve $x-\sin x=2$

The angle x measured in radians, must lie between 2 and 3 Take $\alpha=2.5$

$$f(\alpha) = \cdot 5 - \sin 2.5 = \cdot 5 - \sin 143^\circ 14' = -\cdot 0.97$$
$$f(\alpha) = 1 - \cos 2.5 = 1.801$$

Hence $x = 0.0539$ $b = \alpha + h = 2.5539$

A second approximation gives us $f(b) = -\cdot 00054$ $f'(b) = 1.8322$ $h = \cdot 0002947$ Hence $x = b + h = 2.554195$

58. Complex Roots of Numerical Equations. Recently methods for approximating to the complex as well as the real roots of numerical equations have been perfected. The exposition of these methods is too long for a work like this. See Emory McClintock "A Method for Calculating Simultaneously All the Roots of an Equation", in the *American Journal of Mathematics*, Vol. XVII., pp. 89-110; M. E. Carvallo, *Méthode pratique pour la Résolution numérique complète des Equations algébriques ou transcendantes,* Paris, 1896.

CHAPTER V

THE ALGEBRAIC SOLUTION OF THE CUBIC AND QUADRATIC

59. Solution of the Cubic. There are many different solutions of the general cubic equation

$$b_0x^3 + 3b_1x^2 + 3b_2x + b_3 = 0 \tag{I}$$

The one which we shall give is due to the Italian Mathematician **Tartaglia** and was first published in 1545 by Cardan. Equations I is first transformed into another whose second term is wanting. Putting, as in § 33 $x = \frac{z - b_1}{b_0}$ we get

$$z^3 + 3Hz + G = 0 \tag{II}$$

where $H = b_0b_2 - b_1^2$ and $G = b_0^2b_3 - 3b_0b_1b_2 + 2b_1^3$ To solve equation II, let $z = u + v$ Substituting in II, we get $u^3 + v^3 + 3(uv + H)(u + v) + G = 0$ We are permitted to subject the quantities u and v to a second condition. The most convenient assumption will be

$$uv + H = 0 \tag{III}$$

This yields
$$u^3 + v^3 = -G \tag{IV}$$

Eliminating υ between III and IV, we get $u^3 - \dfrac{H^3}{u^3} = -G$ or $u^6 + Gu^3 = H^3$

The last equation is a quadratic in form. Solving it, we have $u^3 = -\dfrac{G}{2} + \sqrt{\dfrac{G^2}{4} + H^3}$ Then by IV, $\upsilon^3 = -G - u^3 = -\dfrac{G}{2} - \sqrt{\dfrac{G^2}{4} + H^3}$

Since $u = \sqrt[3]{-\dfrac{G}{2} + \sqrt{\dfrac{G^2}{4} + H^3}}$ $\upsilon = \sqrt[3]{-\dfrac{G}{2} - \sqrt{\dfrac{G^2}{4} + H^3}}$ **V**

and $z = u + \upsilon$ we have

$$z = \sqrt[3]{-\frac{G}{2} + \sqrt{\frac{G^2}{4} + H^3}} + \sqrt[3]{-\frac{G}{2} - \sqrt{\frac{G^2}{4} + H^3}} \tag*{\textbf{VI}}$$

The expression for the root of the cubic, given in formula VI is known as **Cardan's formula.**

Since a number has three (3) cube roots, it is evident from cubic equation V that u and υ have each three (3) values. It may seem as if with each value of u we might be able to associate any one of the three values of υ thus obtaining all together nine (9) values for $u+\upsilon$ or z As the cubic has only three (3) roots, this cannot be. Of the nine (9) values, six (6) are excluded by equation III, which u and υ must satisfy. Eliminating υ between $z = u + \upsilon$ and equation III, we get

$$z = u - \frac{H}{u} \tag*{\textbf{VII}}$$

where u has the form given in equation V. Since in expression VII there is only one number u which has triple values, this expression does not involve the difficulties of Cardan's formula. Let the three values of u be $u, u\omega, u\omega^2$ where ω stands for one of the two complex cube roots of unity $-\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{-3}$ Then the three roots of the cubic II are

$$u - \frac{H}{u} \quad u\omega - \frac{H\omega^2}{u} \quad u\omega^2 - \frac{H\omega}{u} \tag*{\textbf{VIII}}$$

Since $z = b_0 x + b_1$ we obtain the roots of the general cubic I by subtracting b_1 from each of the three expressions in cubic equation VIII, and then dividing the three (3) results by b_0

60. Irreducible Case. The general expression for the roots of a quadratic equation with literal coefficients may be used conveniently in solving numerical quadratic equations. For each letter we substitute its numerical value, and then carry out the indicated operations. It is an interesting fact that, in case of the *cubic*, this mode of procedure is not always possible and that the algebraic solution of the cubic is of little practical use in finding the numerical values of the roots.

In § 36 we found that the roots of the cubic are all real when G^2+4H^3 is negative. In the attempt to compute these real roots of the cubic by substituting the values of H and G in the general formula, we encounter the problem to extract the cube root of a complex number. But there exists no convenient arithmetical process of doing this. Nor is there any way of avoiding the complex radicals and of expressing the values of the real roots by real radicals. By the older mathematicians this case when G^2+4H^3 is negative, was called the *"irreducible case"* in the solution of the cubic, the word *"irreducible"* having here a meaning different from that now assigned to it in algebra. See § 123.

61. Solution by Trigonometry. The *"irreducible case"* may be disposed of by expanding the two terms in Cardan's formula into two converging series with the aid of the binomial theorem. The imaginary terms will disappear in the addition of the two series. But it is better to use the following trigonometric method (which is itself inferior, for the purpose of arithmetical computation, to **Horner's method,** § 56).

Let $-\dfrac{G}{2}=r\cos\theta \quad \sqrt{\dfrac{G^2}{4}+H^3}=ir\sin\theta$

We get $u^3=r\left(\cos\theta+i\sin\theta\right) \quad \upsilon^3=r\left(\cos\theta-i\sin\theta\right)$

where $r=\sqrt{-H^3} \quad \cos\theta=\dfrac{-G}{2\sqrt{-H^3}}$

Hence, $u=\sqrt{-H}\left(\cos\dfrac{2n\pi+\theta}{3}+i\sin\dfrac{2n\pi+\theta}{3}\right)$

$$v = \sqrt{-H}\left(\cos\frac{2n\pi+\theta}{3} - i\sin\frac{2n\pi+\theta}{3}\right)$$

and $$z = u + v = 2\sqrt{-H}\cos\frac{2n\pi+\theta}{3}$$

where n takes the values 0, 1, 2

62. Euler's Solution of Quadric. Removing the second term of the quadric

$$b_0x^4 + 4b_1x^3 + 6b_2x^2 + 4b_3x + b_4 = 0 \qquad \textbf{I}$$

we get as in § 34

$$z^4 + 6Hz^2 + 4Gz + b_0^{\,2}I - 3H^2 = 0 \qquad \textbf{II}$$

where $z = b_0x + b_1 \quad H = b_0b_2 - b_1^{\,2} \quad I = b_0b_4 - 4b_1b_3 + 3b_2^{\,2}$

$$G = b_0^{\,2}b_3 - 3b_0b_1b_2 + 2b_1^{\,3}$$

Euler assumes the general expression for a root of equation II to be

$$z = \sqrt{u} + \sqrt{v} + \sqrt{w}$$

Squaring, $z^2 - u - v - w = 2\sqrt{u}\sqrt{v} + 2\sqrt{u}\sqrt{w} + 2\sqrt{v}\sqrt{w}$

Squaring again and simplifying,

$$\begin{aligned}&z^4 - 2z^2(u+v+w) - 8z\sqrt{u}\sqrt{v}\sqrt{w} + (u+v+w)^2 - \\ &-4(uv+uw+vw) = 0\end{aligned}$$

Equating coefficients of this and equation II, we have

$$-3H = u+v+w \quad G = -2\sqrt{u}\sqrt{v}\sqrt{w} \quad \begin{aligned}&(u+v+w)^2 - 4(uv+uw+vw) = \\ &= b_0^{\,2}I - 3H^2\end{aligned}$$

$$\text{or } uv + uw + vw = 3H^2 - \frac{b_0^{\,2}I}{4}$$

But $-(u+v+w)$ $(uv+uw+vw)$ $-uvw$ are the coefficients of a cubic whose roots are u, v, w This cubic, called **"Euler's cubic",** is

$$y^3 + 3Hy^2 + \left(3H^2 - \frac{b_0^{\,2}I}{4}\right)y - \frac{G^2}{4} = 0 \qquad \textbf{III}$$

Let $y = b_0^{\,2}x - H$ and we obtain

$$4b_0^{\,3}x^3 - b_0Ix + J = 0 \qquad \textbf{IV}$$

where $b_0^{\,3}J = b_0^{\,2}IH - 4H^3 - G^2$

Equation IV is called the *reducing cubic* of the quadric. Since $u,\ \upsilon,\ w$ are the three (3) values of y in equation III, we have

$$u = b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_1 \quad \upsilon = b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_2 \quad w = b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_3$$

Hence, $z = \sqrt{b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_1} + \sqrt{b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_2} + \sqrt{b_1^{\,2} - b_0 b_2 + b_0^{\,2} x_3}$ **V**

Or, since $G = -2\sqrt{u}\sqrt{\upsilon}\sqrt{w}$ we may write

$$z = \sqrt{u} + \sqrt{\upsilon} - \frac{G}{2\sqrt{u}\sqrt{\upsilon}} \qquad \textbf{VI}$$

In the expression for z in equation VI each of the radicals may be either + or – Hence z has four (4) values – the four roots of equation II. In equation V there are apparently eight (8) values of z but four (4) of them are ruled out by the relation $2\sqrt{u}\sqrt{\upsilon}\sqrt{w} = -G$

From the above we see that the roots of the quadric are expressed in terms of $u,\ \upsilon,\ w$ The values of the latter are given in terms of the coefficients of the quadric and the three roots $x_1,\ x_2,\ x_3$ of the cubic IV. To solve the quadric by the present method we must, therefore, first solve the *reducing cubic.* There are many other algebraic solutions of the general quadric, but every one of them calls for the solution of an auxiliary equation of the third degree. These cubes are called *resolvents. (διαλυτικοί).*

50. Nature of the Roots of the Quadric. In the study of the nature of the roots of the cubic equation we began in § 35 by deducing the *“equation of squared differences of the roots of the cubic”.* Then, in §36, we used this transformed equation in the discussion of the roots of the given cubic. The same mode of procedure might be adopted in the study of the roots of the quadric equation. But the formation of the *“equation of squared differences of the roots”* is laborious, and we prefer to begin the discussion by applying **Sturm’s Theorem** to the quadric with its second term removed.

If we transform the general quadric

$$b_0 x^4 + 4b_1 x^3 + 6b_2 x^2 + 4b_3 x + b_4 = 0 \qquad \textbf{I}$$

into a new equation, deprived of its second term and with coefficients integral in form, we obtain, as in § 34,

$$y^4+6Hy^2+4Gy+b_0{}^2I-3H^2=0 \qquad \textbf{II}$$

where $y=b_0x+b_1$

$$H\equiv b_0b_2-b_1{}^2$$

$$G\equiv b_0{}^2b_3-3b_0b_1b_2+2b_1{}^3$$

$$I\equiv b_0b_4-4b_1b_3+3b_2{}^2$$

Representing the left member of equation II by $f(y)$ we get $\frac{f'(y)}{4}=y^3+3Hy+G$ and, by division, $f_2(y)=-3Hy^2-3Gy-b_0{}^2I+3H^2$

Before dividing $\frac{1}{4}f'(y)$ by $f_2(y)$ we multiply $\frac{1}{4}f'(y)$ by the positive factor $3H^2$ and we obtain, after dividing the remainder by $b_0{}^2$

$$f_3(y)=\left(b_0{}^2HI-3G^2-12H^3\right)\frac{y}{b_0{}^2}-GI$$

We find it convenient to let $b_0{}^2HI-G^2-4H^3\equiv b_0{}^3J$ Then

$$f_3(y)=(3b_0J-2HI)y-GI$$

Now we multiply $f_2(y)$ by the positive factor $(3b_0J-2HI)^2$ and we obtain, after division, a remainder which, with its sign changed, is equal to $\begin{aligned}&\left(b_0{}^2I-3H^2\right)(3b_0J-2HI)^2+3G^2I(3b_0J-HI)=\\&=b_0{}^2H^2I^3-27b_0{}^2H^2J^2+T\end{aligned}$ where

$$\begin{aligned}T&\equiv\left(9b_0{}^4IJ^2-12b_0{}^3HI^2J+36b_0H^3IJ+9b_0G^2IJ\right)+\\&+\left(3b_0{}^2H^2I^3-3G^2I^2H-12H^4I^2\right)=\\&=3b_0IJ\left(3b_0{}^3J-4b_0{}^2HI+12H^3+3G^2\right)+3b_0{}^3I^2HJ=\\&=3b_0IJ\left(3b_0{}^3J-3b_0{}^2HI+12H^3+3G^2\right)=\\&=3b_0IJ\left(3b_0{}^3J-3b_0{}^3J\right)=0\end{aligned}$$

If the remainder is divided by the positive factor $b_0{}^2H^2$ we obtain

$$f_4(y)=I^3-27J^2$$

We have now all of *Sturm's functions* of equation II (4 assumptions).

(1) *All roots real.* If $\left(I^3-27J^2\right) \succ 0$ $(3b_0J-2HI) \succ 0$ and $H \prec 0$ then, for $y=\infty$ the signs of Sturm's functions are $+++++$ for $y=-\infty$ the signs are $+-+-+$ The excess of variations in the latter case is four; hence all the roots are real.

(2) *All roots imaginary.* If $I^3-27J^2 \succ 0$ and if $H \succ 0$ or else $(3b_0J-2HI) \prec 0$ then the number of variations in signs for $y=-\infty$ is the same as for $y=\infty$ hence there are no real roots.

(3) *Two real roots.* If $I^3-27J^2 \prec 0$ then, no matter what signs H and $(3b_0J-2HI)$ may have, we get always a difference of two variations for $y=\infty$ and $y=-\infty$ hence there are two real roots and two imaginary roots.

(4) *Equal roots.* When $I^3-27J^2=0$ it is evident from the theory of the H.C.F. that there are equal roots. If $f_4(y)$ is the only one of Sturm's functions which vanishes identically, then $f_3(y)$ is the H.C.F. in y and there are two roots equal to each other. If $f_3(y)$ is identically zero (0), which happens when $I=0$ and $J=0$ or when $G=0$ and $3b_0J=2HI$ then three roots are equal to each other or there are two distinct pairs of double roots. That is, if $I=0$ and $J=0$ we get from the equation defining J the relation $G^2+4H^3=0$ which makes $f_2(y)$ a perfect square. Hence three roots are equal. When $G=0$ and $3b_0J=2HI$ it follows that ${b_0}^2I=12H^2$ and $f_2(y)$ is readily seen to be composed of two unequal factors in y indicating the existence of two distinct pairs of equal roots. If we have $I=0$ $J=0$ and $H=0$ then it follows that $G=0$ and $f_2(y)=0$ Hence $f'(x)=y^3$ and all the roots are equal.

This discussion of equation II applies also to equation I, representing the general quadric; for, since $y=b_0x+b_1$ the values of x

are real, imaginary, or multiple values, according as the values of y are real, imaginary, or multiple values.

NEWTON'S METHOD OF APPROXIMATING TO THE INCOMMENSURABLE ROOTS OF AN EQUATION

204. The method proposed by **Newton** for approximating to the incommensurable roots which may still exist in an equation, after the integral roots have been remove by the method of divisors, requires, like all other approximating methods, that we know the intervals in which the roots are situated. It requires, moreover, that before commencing the approximation to any root, we render to interval so narrow, that the extreme limits of it may not differ by more than $\frac{1}{10}$ in which case either limit must be within the fraction $\frac{1}{10}$ of the value of the root. Call the initial value, thus obtained, x' and its difference from the true root δ then, in the proposed equations $f(x)=0$ we have $x=x'+\delta$ and, consequently,

$$f(x)=f(x'+\delta)=f(x')+f_1(x')\delta+\frac{f_2(x')}{2}\delta^2+\&c.=0$$

and, since δ is less than $\frac{1}{10}$ δ^2 must be less than $\frac{1}{100}$ δ^3 less than $\frac{1}{1000}$ &c. Hence, rejecting the terms into which these diminishing factors enter, we have, for a first approximation to the value of the correction δ, the expression

$$\delta=-\frac{f(x')}{f_1(x')}$$

which will give the value true to two places (2) of decimals: adding, therefore, this approximate correction to x' we obtain a nearer value x'' to the root, the error δ' being below $\frac{1}{100}$

For a second approximation, we put $x = x'' + \delta'$ then, proceeding as before, we have $\delta' = -\frac{f(x'')}{f_1(x'')}$ which will usually give the value of the correction, as far as four places (4) of decimals, and this correction applied to x'' will give the more correct value x''' for x being the true value, as far as about four (4) decimals; and, by repeating the operation, we shall get a new value, true to about eight (8) decimals, and so on.

The following is the example chosen by Newton to illustrate his method, viz.

$$f(x) = x^3 - 2x - 5 = 0 \quad \therefore f_1(x) = 3x^2 - 2$$

The root of this equation lies between 2 and 3 To narrow these limits, diminish the roots of the transformed in $x-3$ by $0,5$ and will shall find no change of sign in the final term; hence the root is between 2 and $2,5$ Diminish the roots of this transformed by $0,4$ and still the final sign is preserved; hence the root is between 2 and 2.1 so that the first two figures of it must be 2.0 that is,

$$x = 2,0 + \delta$$

$$\text{Also } \delta = -\frac{f(2.0)}{f_1(2.0)} = -\frac{-1}{10} = 0.1$$

$$\therefore x = 2.1 + \delta'$$

$$\delta' = -\frac{f(2.1)}{f_1(2.1)} = -\frac{0.061}{11.23} = -0.0054$$

$$\therefore x = 2.0946$$

$$\delta'' = -\frac{f(2.0946)}{f_1(2.0946)} = -\frac{0.0005416}{11.16205} = -0.00004852$$

$$\therefore x = 2.09455148$$

In this particular example the approximation is very rapid; this arises from the circumstances that, in the expressions for δ' δ'' &c. the numerators are very small when compared with the denominators; such, however, will not be the case, when the root, to which we are approaching, differs but little from another root; because, as the roots approach to equality, the expression $f_1(x)$ when the value of one of these roots is put for x approaches to zero (98); and hence the denominators of the foregoing fractions will be very small, as well as the numerators. In such a case, too, the terms rejected in the values of δ' δ'' &c. might exceed in magnitude those preserved, and thus no approximation to the true correction would be obtained. These imperfections in Newton's process render its application unsafe, when the root sought differs by only a small decimal form any of the other real roots, unless, indeed, at each approximation, we test the value obtained, by actually substituting it in the proposed equation.

As an illustration, let the equation $x^3 - 7x + 7 = 0$ be proposed.

After a few trials, a root is found to lie between 1.3 and 1.4 and to be nearer to 1.4 than to 1.3 Let us assume that $x = 1.4 + \delta$ Then we have

$$\delta = -\frac{f(1.4)}{f_1(1.4)} = -\frac{0.056}{1.12} = -0.05$$

$$\therefore x = 1.35 + \delta'$$

To verify this approximation, let 1.35 and 1.36 be separately put for x in the proposed equation, the results are

$$\text{for } x = 1.35 \quad f(x) = +0.010375$$

$$\text{for } x = 1.36 \quad f(x) = -0.004544$$

which, being of contrary signs, shows that our approximation is correct.

For a second approximation we have

$$\delta'' = -\frac{f(1.35)}{f_1(1.35)} = \frac{0.010375}{1.5325} = -0.0068$$

$$\therefore x = 1.3568$$

The root-figures after the comma are negative, since in the second transformation the root is overstepped; and by this manner we always may proceed whenever, either by mistake or otherwise, too great a root-figure is employed in any step. By actually subtracting the negative portion of the root from the positive we find the following equation

$$x = 2.0945514815423$$

On account of the smallness of the correction by which each trial-divisor is converted into a true divisor the preceding example is peculiarly favorable to the method of Newton: yet the operation conducted as above, and by which a single figure only is determined at each step, is even in this case far more brief and simple than the corresponding process of Newton, though several figures of the root are furnished at each step. This will be more clearly seen by comparing the foregoing work with the details of the calculation as given by **Fourier**, in pages 212-216 of the *Analyse des Equations*, according to what he considers an improved mode of conducting the Newtonian operation.

SECTION X

THEORY OF LAGRANGE ON THE GENERAL SOLUTION OF EQUATIONS

179. A remarkable application of the theory of symmetrical functions is that made by **Lagrange** to the general solution of equations; by that means he solves the general equations of the first four degrees, by a uniform process, and one which includes all others that have been proposed for that purpose, the common relation of which to one another is thus made apparent.

It consists in employing an auxiliary equation, called a *reducing equation*, whose root is of the form

$$x_1 + \alpha x_2 + \alpha^2 x_3 + \ldots + \alpha^{n-1} x_n$$

denoting by $x_1, x_2, \ldots x_n$ the n roots of the proposed equation, and by α one of the n^{th} roots of unity; and the principle on which it is based is as follows. Let y be the unknown quantity in the reducing equation, and let

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

$\alpha_1, \alpha_2, \ldots \alpha_n$ denoting certain constant quantities; then if $n-1$ values of y and suitable values of the constants $\alpha_1, \alpha_2, \ldots \alpha_n$ can be found, so that we may have $n-1$ simple equations; those, together with the equation

$$-p_1 = x_1 + x_2 + \ldots + x_n$$

will enable us to determine the n roots.

Now, supposing the constants in the value of y to preserve an invariable order $\alpha_1, \alpha_2, \&c.$ since the number of ways in which the n roots may be combined with them to form the expression $\alpha_1 x_1 + \alpha_2 x_2 + \&c.$ is the same as the number of permutations of n things taken all together; therefore the expression for y will have $n(n-1)\ldots 3\cdot 2\cdot 1$ values, and the equation for determining y will rise to the same number of dimensions, or will be of a degree higher than that of the proposed equation; hence the method will be of no use, unless such values can be assumed for the constants $\alpha_1, \alpha_2, \ldots \alpha_n$ as shall make the solution of the equation in y depend upon that of an equation at most of $n-1$ dimensions. Now this may be done (at least when n does not exceed 4) by taking the n^{th} roots of unity $\alpha^0, \alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$ for $\alpha_1, \alpha_2, \ldots \alpha_n$ so that

$$y = \alpha^0 x_1 + \alpha x_2 + \ldots + \alpha^{r-1} x_r + \alpha^r x_{r+1} + \ldots + \alpha^{n-1} x_n$$

For, in the first place, with this assumptions, the reducing equation will contain only powers of y which are multiples of n For, since $\alpha^n = 1$ we have

$$\alpha^{n-r} y = \alpha^{n-r} x_1 + \alpha^{n-r+1} x_2 + \ldots + x_{r+1} + \alpha x_{r+2} + \ldots + \alpha^{n-r-1} x_n$$

$$\text{or } \alpha^{n-r} y = \alpha^0 x_{r+1} + \alpha x_{r+2} + \ldots + \alpha^{n-1} x_r$$

which is the same result as if we had interchanged x_1 and x_{r+1} , x_2 and x_{r+2} ,&c. so that if y be a root of the reducing equation, $\alpha^{n-r} y$ is also a root; therefore the reducing equation, since it remains unaltered when $\alpha^{n-r} y$ is written for y contains only powers of y which are multiples of n If therefore we make $y^n = z$ we shall have a reducing equation in z of only $1\cdot 2\cdot 3\ldots(n-1)$ dimensions, whose roots will be the different values of z which result from the permutations of the $n-1$ roots $x_2, x_3, \ldots x_n$ among themselves. We shall now have, expanding and reducing

$$z = y^n = u_0 + u_1\alpha + u_2\alpha^2 + \ldots + u_{n-1}\alpha^{n-1}$$

in which $u_0, u_1, u_2, \ldots u_{n-1}$ are determinate functions of the roots, which will be invariable for the simultaneous changes of x_1 into x_{r+1} x_2 into x_{r+2},&c. since $z = \left(\alpha^r y\right)^n$ and when their values are known in terms of the coefficients of the proposed equation, we shall immediately know the values of the roots. For let $z_0, z_1, z_2, \ldots z_{n-1}$ be the different values of z when $1, \alpha, \beta, \gamma \ldots . \lambda$ the roots of $y^n - 1 = 0$ are substituted for α Then since $y = \sqrt[n]{z}$ we have

$$x_1 + x_2 + \ldots + x_n = \sqrt[n]{z_0}$$

$$x_1 + \alpha x_2 + \ldots + \alpha^{n-1} x_n = \sqrt[n]{z_1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots$$

$$x_1 + \lambda x_2 + \ldots + \lambda^{n-1} x_n = \sqrt[n]{z_{n-1}}$$

therefore, adding, and taking account of the properties of the sums of the powers of $1, \alpha, \beta, \gamma$,&c. (Art. 154), we get

$$nx_1 = \sqrt[n]{z_0} + \sqrt[n]{z_1} + \ldots + \sqrt[n]{z_{n-1}}$$

Again, multiplying the above system of equations respectively by 1, $\alpha^{n-1}, \beta^{n-1}, \ldots . \lambda^{n-1}$ we get

$$nx_2 = \sqrt[n]{z_0} + \alpha^{n-1}\sqrt[n]{z_1} + \beta^{n-1}\sqrt[n]{z_2} + \ldots + \lambda^{n-1}\sqrt[n]{z_{n-1}}$$

and so on for the rest. Hence, since $-p_1 = \sqrt[n]{z_0}$ and

$$\therefore \left(-p_1\right)^n = z_0 = u_0 + u_1 + \ldots + u_{n-1}$$

the problem is reduced to finding the values of $u_1, u_2, \ldots u_{n-1}$

180. The reducing equation of $n-1$ dimensions, which has for its roots $z_1, z_2, \ldots z_{n-1}$ the quantities by which the roots of the proposed equation have just been expressed, will require, for the determination of its coefficients, the solution of an equation of $1\cdot 2\cdot 3\ldots(n-2)$ dimensions.

For if in the equation

$$z=\left(x_1+\alpha x_2+\alpha^2 x_3+\ldots+\alpha^{n-1}x_n\right)^n$$

we replace α by each of its powers $\alpha^2,\alpha^3,\ldots\alpha^{n-1}$ and denote as before the corresponding values of z by $z_2,z_3,\ldots z_{n-1}$ and call the above value of z,z_1 we get

$$z_1=\left(x_1+\alpha x_2+\alpha^2 x_3+\alpha^3 x_4+\ldots+\alpha^{n-1}x_n\right)^n$$

$$z_2=\left(x_1+\ldots+\alpha^2 x_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)^n$$

$$z_3=\left(x_1+\ldots\ldots\ldots\ldots+\alpha^3 x_2+\ldots\ldots\ldots\ldots\ldots\right)^n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z_{n-1}=\left(x_1+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\alpha^{n-1}x_2\right)^n$$

The equation z_3 is used as equation (I).

$$z_3=\left(x_1+\ldots\ldots\ldots\ldots+\alpha^3 x_2+\ldots\ldots\ldots\ldots\ldots\right)^n \qquad \textbf{(I)}$$

In these expressions x_2 occupies, successively, the second, third, &c., n^{th} place; that is, all the places it can occupy subject to the condition that x_1 always stands first. If therefore in each of these expressions we make all possible permutations of $x_3,x_4,\ldots x_n$ {the number of which is $1\cdot 2\cdot 3\ldots(n-2)$} without altering the place of x_1 or x_2 we shall from each obtain $1\cdot 2\cdot 3\ldots(n-2)$ values of z and therefore on the whole we shall obtain the $1\cdot 2\cdot 3\ldots(n-1)$ values which z admits of from the permutations of the $n-1$ roots $x_2,x_3,\ldots x_n$

Now let

$$z^{n-1}+q_1^{z^{n-2}}+q_2^{z^{n-3}}+\ldots+q_{n-1}=0 \qquad \textbf{(II)}$$

be the equation which has the above quantities z_1, z_2 and z_{n-1} for its roots; then the coefficients $q_1, q_2, \&c.$ will depend for their determination upon an equation whose degree is $1 \cdot 2 \cdot 3 \ldots (n-2)$

For suppose that, by causing another root x_2 to stand first in the expressions (I) we form another system of values of $z_1, z_2, \ldots z_{n-1}$ and another equation of which they are the roots, similar to the expression (II) viz.

$$z^{n-1} + q_1'^{z^{n-2}} + \ldots + q_{n-1}' = 0 \text{ and so on;}$$

and let k be the number of such equations necessary to be formed in order to furnish al the values of z Then the first members of these equations multiplied together will be of the degree $k(n-1)$ and will be the final equation whose roots are all the values of z and whose coefficients, being symmetrical functions of the roots of the proposed, are capable of being expressed rationally by its coefficients. Hence, as the degree of the final equation is equally expressed by $k(n-1)$ or $1 \cdot 2 \cdot 3 \ldots (n-1)$ we find $k = 1 \cdot 2 \cdot 3 \ldots (n-2)$ Also since there will be k equations similar to the expression (II), there will be the same number of values of q_1 and the coefficients, consequently, of the expression (II) will depend for their determination upon an equation whose degree is $1 \cdot 2 \cdot 3 \ldots (n-2)$

To form the equation for determining q_1 we must divide the final equation in z by the first member of the expression (II) which is one of its factors, and equate to zero (0) the $n-1$ terms of the remainder; then $n-2$ of these equations will serve to determine $q_2, q_3, \&c.$ in terms of q_1 and substituting the values of $q_2, q_3, \&c.$ in the remaining equation, we shall have an equation in q_1 whose degree will be equal to $1 \cdot 2 \cdot 3 \ldots (n-2)$ If we know one system of values of the coefficients $q_1, q_2, \&c.$ and if we can solve the corresponding equation in z of $n-1$ dimensions, so as to obtain the values of $z_1, z_2, \&c., z_{n-1}$ then the solution of the proposed equation follows, as has been shown. The result of the entire process would be that the equation of $1\cdot 2\cdot 3\ldots(n-1)$ dimensions, having for its roots all the values of z would be resolved into $1\cdot 2\cdot 3\ldots(n-2)$ factors of $n-1$ dimensions, by means of a single equation whose degree equals $1\cdot 2\cdot 3\ldots(n-2)$

181. When n is a composite number, the above general method admits of simplifications. For let n have a divisor m so that $n=mp$ where m is not greater than p and let α be a root of $y^m-1=0$ then since $\alpha^m=1, \alpha^{m+1}=\alpha,\ \alpha^{m+2}=\alpha^2,\&c.,\ \alpha^{2m}=1,\ \alpha^{2m+1}=\alpha,\&c.$ we have

$$y=x_1+\alpha x_2+\alpha^2 x_3+\ldots+\alpha^{n-1}x_n=X_1+\alpha X_2+\alpha^2 X_3+\ldots+\alpha^{m-1}X_m$$

where $X_r=x_r+x_{m+r}+x_{2m+r}+\ldots+x_{n-m+r}$ and consists of p roots;

$$\therefore z=y^n=u_0+u_1\alpha+u_2\alpha^2+\ldots+u_{m-1}\alpha^{m-1}$$

where $u_0, u_1,\&c.$ are known functions of $X_1, X_2\ \&c.$ and when they are found in terms of the coefficients of the proposed equation, we shall be able to determine immediately the values of $X_1, X_2\ \&c.$ as before. To deduce the value of the primitive roots $x_1, x_2, x_3, \ldots x_n$ we must regard separately those which compose each of the quantities $X_1, X_2\ \&c.$ as the roots of an equation of p dimensions. Thus let the roots whose sum is X_1 be those of the equation

$$x^p-X_1x^{p-1}+Lx^{p-2}-Mx^{p-3}+\ldots=0 \qquad \textbf{(I)}$$

where $L,\ M,\&c.$ are unknown; then the first member of this equation is a divisor of the first member of the proposed, since all its roots belong to the latter. Hence, effecting the division and equating to zero the coefficients of $x^{p-1}, x^{p-2},\&c.$ in the remainder, we shall have p equations in $X_1,\ L,\ M,\&c.$ of which the $p-1$ first will give the values of $L,\ M,\&c.$ in terms of X_1 It will then remain to solve the equation (I) so determined of p dimensions. Similarly, substituting the value of X_2

in place of that of X_1 we shall have an equation giving the next group of roots $x_2, x_{m+2}, \&c.$ and so on.

182. In this case, that is when n is a composite number and $= mp$ where m is a prime number, the formation of the reducing equation will require the solution of an equation of only $\dfrac{1\cdot 2\cdot 3\ldots n}{(m-1)m(1\cdot 2\cdot 3\ldots p)^m}$ dimensions.

For since $y = X_1 + \alpha X_2 + \alpha^2 X_3 + \ldots + \alpha^{m-1} X_m$ if everyone of the roots found in $X_1, X_2, \&c.$ were affected with a different coefficient, since there are n of those roots, the number of distinct values, which y would be capable of acquiring from their permutations, would be $1\cdot 2\cdot 3\ldots n$ But on account of the p roots in each group having the same coefficient, the number of values of y is diminished. Let μ be this number; then supposing all the roots in the group X_1 to have distinct coefficients and so to furnish $1\cdot 2\cdot 3\ldots p$ permutations, the number of values of y would be increased to $1\cdot 2\cdot 3\ldots p \times \mu$ Next, if all the roots in X_2 received distinct coefficients, the number of values of y would become $(1\cdot 2\cdot 3\ldots p)^2 \times \mu$ and so on; so that if every root in every group had a distinct coefficient, the number of values of y would be $(1\cdot 2\cdot 3\ldots p)^m \times \mu$ which, as we have seen above, is equal to $1\cdot 2\cdot 3\ldots n$ But, as shown in Art. 179, the number of values of z where

$$z = \left(X_1 + \alpha X_2 + \alpha^2 X_3 + \ldots + \alpha^{m-1} X_m\right)^m \qquad \textbf{(I)}$$

is $\dfrac{1}{m} th$ of the number of values of y and therefore is equal to $\mu \div m$ Also, if as before we denote by z_1 the value of z in equation (I), and by $z_2, z_3, \ldots z_{m-1}$ the values which z assumes when α is successively replaced by its powers $\alpha^2, \alpha^3, \ldots \alpha^{m-1}$ we may form an equation which has the quantities $z_1, z_2, \ldots z_{m-1}$ for its roots, viz.

$$z^{m-1} + q_1^{z^{m-2}} + \&c. + q_{m-1} = 0 \qquad \textbf{(II)}$$

and next by causing another of the quantities X_2 to stand first in the value of y we may form another system of values of $z_1, z_2, \ldots z_{m-1}$ and another equation similar to the equation (II), viz.

$$z^{m-1} + q_1'^{z^{m-2}} + \ldots + q'_{m-1} = 0 \text{ and so on.}$$

Let k denote the number of such equations necessary to be formed, in order to furnish all the values of z Then the first members of these equations multiplied together will be the final equation in z and its degree will be $k(m-1)$ which is the same as the number of values of z and equals $\mu \div m$ Therefore, substituting for μ its value, we find

$$k = \frac{1 \cdot 2 \cdot 3 \ldots n}{m(m-1)(1 \cdot 2 \cdot 3 \ldots p)^m}$$

Since, therefore, there will be k equations similar to the equation (II), there will be the same number of values of the coefficient q_1 and the coefficients, consequently, of equation (II) will depend for their determination upon an equation whose degree is equal to the value of k above written.

183. This is the point to which the investigation of the algebraical solution of equations was brought by Lagrange, and where is still remaining at the present time. The method leads to the solution of equations of the 3rd and 4th degree, as we proceed to show; but for the equation of the 5th degree

$$x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$$

if, taking α for an imaginary root of $x^5 - 1 = 0$ we assume

$$z = \left(x_1 + \alpha x_2 + \alpha^2 x_3 + \alpha^3 x_4 + \alpha^4 x_5\right)^5$$

we may form the equation with determined coefficients whose roots are all the values of z and whose degree will $= 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and its first member will be capable of being resolved into 2×3 or 6 biquadrate equations of the form

$$z^4 + q_1 z^3 + q_2 z^2 + q_3 z + q_4 = 0$$

where each of the coefficients $q_2, q_2, \&c.$ admits of 2×3 values, for different permutations of the roots; and will therefore depend upon the solution of an equation of the 6th degree. So that by this process the solution of an equation of the 5th degree will necessarily involve the solution of another equation of a higher degree than its own.

Example 1

$$x^3 - px^2 + qx - r = 0$$

Let the roots be α, b, c and let $y = \alpha + \alpha b + \alpha^2 c$

$$\therefore z = y^3 = \alpha^3 + b^3 + c^3 + 6\alpha bc + 3\left(\alpha^2 b + b^2 c + c^2 \alpha\right)\alpha$$
$$+3\left(\alpha^2 c + b^2 \alpha + c^2 b\right)\alpha^2 = u_0 + u_1 \alpha + u_2 \alpha^2$$

But u_1, u_2 are roots of the quadratic $u^2 - (u_1 + u_2)u + u_1 u_2 = 0$ and

$$u_1 + u_2 = 3\sum\left(\alpha^2 b\right) = 3pq - 9r \text{ (Art. 159)},$$

$$u_1 u_2 = 9\left\{\alpha bc S_3 + \sum\left(\alpha^3 b^3\right) + 3\alpha^2 b^2 c^2\right\} =$$
$$= 9q^3 + 9\left(p^3 - 6pq\right)r + 81r^2$$

Hence, u_1, u_2 are known, and $\therefore u^0 = p^3 - (u_1 + u_2)$ is known.

Hence, denoting by z_1, z_2 the values of z when α and α^2 are respectively written for α we have

$$\alpha + b + c = p$$

$$\alpha + \alpha b + \alpha^2 c = \sqrt[3]{z_1}$$

$$\alpha + \alpha^2 b + \alpha c = \sqrt[3]{z_2}$$

from which we obtain the values of α b and c viz.

$$\alpha = \frac{1}{3}\left(p + \sqrt[3]{z_1} + \sqrt[3]{z_2}\right)$$

$$b = \frac{1}{3}\left(p + \alpha^2 \square \sqrt[3]{z_1} + \alpha \square \sqrt[3]{z_2}\right)$$

$$c = \frac{1}{3}\left(p + \alpha \square \sqrt[3]{z_1} + \alpha^2 \square \sqrt[3]{z_2}\right)$$

Example 2

$$x^4 - px^3 + qx^2 - rx + s = 0$$

Since $4 = 2 \cdot 2$ let α be a root of $y^2 - 1 = 0$ so that $\alpha^2 = 1$ then $y = x_1 + \alpha x_2 + x_3 + \alpha x_4 = X_1 + \alpha X_2$ If $X_1 = x_1 + x_3$ $X_2 = x_2 + x$ we shall have:

$$\therefore z = y^2 = u_0 + \alpha u_1$$

where $u_0 = X_1^2 + X_2^2$ $u_1 = 2X_1X_2$ and $u_0 + u_1 = z_0 = p^2$

Hence $u_1 = 2(x_1 + x_3)(x_2 + x_4)$ by interchanging the roots among themselves, will admit the two other values

$$2(x_1 + x_2)(x_3 + x_4) \text{ and } 2(x_1 + x_4)(x_2 + x_3)$$

and will therefore be a root of an equation of the form

$$u_1^3 - Mu_1^2 + Nu_1 - P = 0$$

The coefficients being symmetrical functions of x_1, x_2, x_3, x_4 and consequently assignable in terms of p, q, r, s It is easily seen that if we make $u_1 = 2q - 2u$ we shall have an equation in u whose roots are

$$x_1x_3 + x_2x_4 \quad x_1x_2 + x_3x_4 \quad x_1x_4 + x_2x_3$$

and the transformed equation is (Art. 162)

$$u^3 - qu^2 + (pr - 4s)u - (p^2 - 4q)s - r^2 = 0$$

Let u' be a root of this equation, then $u_1 = 2q - 2u'$ Hence, making $\alpha = -1$ we have the following equations:

$$z_1 = u_0 - u_1 = p^2 - 2u_1 = p^2 - 4q + 4u'$$

$$\therefore X_1 + X_2 = p \quad X_1 - X_2 = \sqrt{z_1}$$

$$\therefore X_1 = \frac{1}{2}\left(p + \sqrt{z_1}\right) \quad \therefore X_2 = \frac{1}{2}\left(p - \sqrt{z_1}\right)$$

Hence, x_1, x_3 may be regarded as roots of a quadratic

$$x^2 - X_1x + L = 0$$

dividing the proposed by this, and putting the first term of the remainder equal to zero (0), we find

$$L=\frac{X_1^3-pX_1^2+qX_1-r}{2X_1-p}$$

Therefore, x_1,x_3 are known; and x_2,x_4 will result from the same formulae by interchanging X_1 and X_2 or by changing the sign of the radical $\sqrt{z_1}$

Example 3

$$\frac{x^n-1}{x-1}=0 \quad n \text{ being a prime number.}$$

If r be one of the roots, and α be a primitive root of the prime number n (that is, a number whose several powers from 1 to $n-1$ when divided by n leave different remainders) it is proved (Art. 80) that all the roots of this equation may be represented by

$$r,\ r^{\alpha},r^{\alpha^2},r^{\alpha^3},\ldots r^{\alpha^{n-2}}$$

$$\text{Let } y=r+\alpha r^{\alpha}+\alpha^2r^{\alpha^2}+\ldots+\alpha^{n-2}r^{\alpha^{n-2}}$$

α being a root of the equation $y^{n-1}-1=0$ Therefore, observing that $\alpha^{n-1}=1$ and $r^n=1$ we have the following equation:

$$z=y^{n-1}=u_0+\alpha u_1+\alpha^2u_2+\ldots+\alpha^{n-2}u_{n-2} \qquad \textbf{(I)}$$

$u_0,u_1,\&c.$ being rational and integral functions of r which do not change by the substitution of $r^{\alpha},r^{\alpha^2},r^{\alpha^3},\&c.$ in the place of r For these quantities, regarded as functions of $x_1,x_2,x_3,\&c.$ do not alter by the simultaneous changes of x_1 into x_2 , x_2 into x_3 &c. nor by the simultaneous changes of x_1 into x_3 , x_2 into x_4 &c. to which correspond the changes of r into r^{α} and r^{α} into r^{α^2} &c.

Now every rational and integral function of r in which $r^n=1$ may be reduced to the form

$$A+Br+Cr^2+Dr^3+\ldots+Nr^{n-1}$$

the coefficients $A,B,C,\ldots N$ being given quantities independent of r or, since in this case the powers $r,\ r^2,r^3,\ldots r^{n-1}$ may be represented, although in a different order, by $r,\ r^{\alpha}, r^{\alpha^2}, \dots r^{\alpha^{n-2}}$ we may reduce every rational function of r to the form

$$A + Br + Cr^{\alpha} + Dr^{\alpha^2} + \dots + Nr^{\alpha^{n-2}}$$

Therefore, if this function is such that it remains unaltered when r is changed into r^{α} it follows that the new form

$$A + Br^{\alpha} + Cr^{\alpha^2} + Dr^{\alpha^3} + \dots + Nr$$

coincides with the preceding;

$$\therefore B = C,\ C = D,\ D = E,\ \ \&c.,\ N = B$$

and therefore the function is reduced to the form

$$A + B\left(r + r^{\alpha} + r^{\alpha^2} + \dots + r^{\alpha^{n-2}}\right) \text{ or } A - B$$

since the sum of the roots is equal to -1 $(= -1)$. Hence each of the quantities $u_0, u_1, u_2, \&c.$ will be of the form $A - B$ and its value will be found by the actual development of $z = y^{n-1}$ so that we have the case where the values of $u_0, u_1, u_2, \&c.$ are known immediately, without depending upon the solution of any equation. Hence if we denote by $1, \alpha, \beta, \gamma, \&c.$ the $n-1$ roots of the equation $x^{n-1} - 1 = 0$ and by $z_0, z_1, z_2, \&c.$ the value of z answering to the substitution of these roots in the place of α in equation (I), we shall have, as in the former cases,

$$r = \frac{\sqrt[n-1]{z_0} + \sqrt[n-1]{z_1} + \sqrt[n-1]{z_2} + \dots + \sqrt[n-1]{z_{n-1}}}{n-1}$$

an expression for one of the roots of the equation $x^n - 1 = 0$ and the other roots are $r^2, r^3, \&c.$

Thus the solution of $x^n - 1 = 0$ is reduced to that of the inferior equation $y^{n-1} - 1 = 0$ of which $1, \alpha, \beta, \gamma, \&c.$ are the roots; also since $n-1$ is a composite number, the determination of $\alpha, \beta, \gamma, \&c.$ will not require the solution of an equation of a higher degree than the greatest prime number in $n-1$ that is, the solution of $x^n - 1 = 0$ (n prime), may be made to depend upon the solution of equations whose degrees do not exceed the greatest prime number which is a divisor of $n-1$

Example 4

$$x^5 - 1 = 0$$

The least primitive root of 5 is 2 For the powers of 2 from 1 to 4 when divided by 5 leave remainders $2, 4, 3, 1$

$$\therefore y = r + \alpha r^2 + \alpha^2 r^4 + \alpha^3 r^3$$

Also $\alpha^4 = 1$ $r^5 = 1$ and $r + r^2 + r^4 + r^3 = -1$

$$\therefore z = y^4 = -1 + 4\alpha + 14\alpha^2 - 16\alpha^3$$

But the four (4) roots of $y^4 - 1 = 0$ are $1,\ -1,\ \sqrt{-1},\ -\sqrt{-1}$

$$\therefore z_0 = 1,\ z_1 = 25,\ z_2 = -15 + 20\sqrt{-1},\ z_3 = -15 - 20\sqrt{-1}$$

$$\therefore x = \frac{1}{4}\left(-1 + \sqrt{5} + \sqrt[4]{-15 + 20\sqrt{-1}} + \sqrt[4]{-15 - 20\sqrt{-1}}\right)$$

CHAPTER VI

SOLUTION OF BINOMIAL EQUATIONS AND RECIPROCAL EQUATIONS

63. The Binomial Equation

Let the equation

$$x^n - \alpha = 0$$

where α is either real or complex, may be solved trigonometrically as follows. Let

$$x^n = \alpha = r\left\{\cos\left(2\kappa\pi + \theta\right) + i\sin\left(2\kappa\pi + \theta\right)\right\}$$

where κ may assume any integral value. Then, by **De Moivre's Theorem** we have: $x = \sqrt[n]{r}\left\{\cos\frac{2\kappa\pi + \theta}{n} + i\sin\frac{2\kappa\pi + \theta}{n}\right\}$

By assigning to κ any n consecutive integral values, we obtain n values for x and no more than n since the n values recur in periods.

It is readily seen that the roots are all complex when α is a complex number. For, to obtain a real root $\frac{2\kappa\pi + \theta}{n}$ must be zero (0) or a multiple of π that is $2\kappa\pi + \theta$ will be zero (0) or a multiple of π Hence α itself must be real, which is contrary to supposition.

When

$$\alpha = +1$$

then

$$x^n = 1 = \cos 2\kappa\pi + i\sin 2\kappa\pi \text{ and } x = \cos\frac{2\kappa\pi}{n} + i\sin\frac{2\kappa\pi}{n}$$

where κ may be assigned the values $0,\ 1,\ldots(n-1)$ If n is odd, then $\kappa = 0$ is the only value of κ which yields a real root, viz. $x = 1$ If n is even, then only the values $\kappa = 0$ and $\kappa = \frac{n}{2}$ yield real roots, viz. $x = 1$ and $x = -1$

When $\alpha = -1$ then

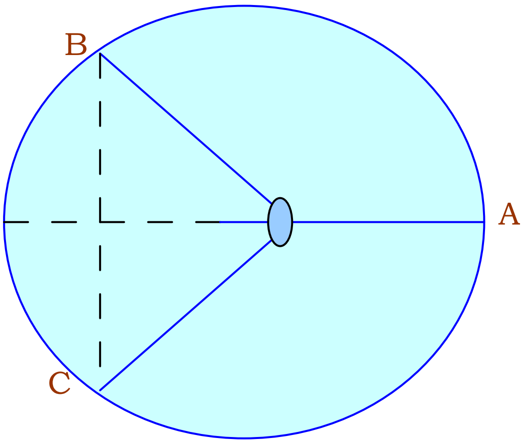
$$x^n = -1 = \cos(2\kappa+1)\pi + i\sin(2\kappa+1)\pi$$

where κ may take the values $0,\ 1,\ldots(n-1)$

Then $x = \cos\frac{(2\kappa+1)\pi}{n} + i\sin\frac{(2\kappa+1)\pi}{n}$ There can be no real roots, unless $\frac{2\kappa+1}{n}$ is an integer, and therefore n an odd number. If $n = 2\kappa+1$ that is $\kappa = \frac{n-1}{2}$ we obtain the real root $x = -1$

64. Geometrical Interpretation of the Roots of $x^n = \alpha$

The n roots may be represented graphically in the **Wessel's Diagram** (§22) by n lines drawn from the centre of a circle of radius $\sqrt[n]{r}$ to points on its circumference and dividing the perigon at the centre into equal angles of $\frac{2\pi}{n}$ radians. Thus, let $n = 3$ and $r = 1$



The three (3) cube roots of unity are seen from the equation (I) §63, to be

$$1,\ -\frac{1}{2}+\frac{1}{2}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3}$$

They are represented, respectively, by the lines *OA*, *OB*, *OC* These lines make with each other angles of $\frac{2}{3}\pi$ radians or 120° The circumference is divided into 3 equal parts. In the general case the circumference is divided into n equal parts. Hence the theory of the roots of unity is closely allied with the problem of inscribing regular polygons in a circle or the theory of the *Division of the Circle.* This subject has been worked out mainly by **C. F. Gauss**, 1801, and will be treated more fully in Chapter XVII under the head of *Cyclotomic Equations.*